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## Dispersion and collapse in stochastic velocity fields on a cylinder

**Abstract** The dynamics of fluid particles on cylindrical manifolds is investigated. The velocity field is obtained by generalizing the isotropic Kraichnan ensemble, and is therefore Gaussian and decorrelated in time. The degree of compressibility is such that when the radius of the cylinder tends to infinity the fluid particles separate in an explosive way. Nevertheless, when the radius is finite the transition probability of the two-particle separation converges to an invariant measure. This behavior is due to the large-scale compressibility generated by the compactification of one dimension of the space.

**Keywords** Turbulence · Lagrangian trajectories · Kraichnan ensemble · Cylindrical manifolds

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## 1 Introduction

Many physical systems display a strong dependence on the space dimensionality, the best known example being given by phase transitions in equilibrium statistical physics. As for non-equilibrium systems, hydrodynamic turbulence shows a remarkable dependence on the space dimension as well. In three dimensions, the kinetic energy flows from large to small scales in the form of a Kolmogorov–Richardson cascade. Conversely, in two dimensions, energy is transferred upscale at a constant rate, in an inverse cascade process [18]. Additionally, three-dimensional turbulence is characterized by a breakdown of scaling invariance and small-scale intermittency [13], whereas the inverse cascade is apparently self-similar [2] and even shows some intriguing signatures of conformal invariance [1].

These observations have spurred the search of a critical dimension between  $d = 2$  and  $d = 3$  in the hope that it could provide a starting point for an analytical attack of three- or two-dimensional turbulence, or both. This approach has been mainly applied to simplified models of turbulence, such as EDQNM [12], the shell model [16], or a model obtained by generalizing the form of the two-dimensional stream function [24]. In those studies, the spatial dimension has been most conveniently reduced to a formal parameter that could take arbitrary values. The approach undertaken in the present paper differs from previous work at least in two important aspects. First, we shall consider a geometrical, rather than formal, way of looking in between dimensions. Namely, we shall study the dynamics of fluid particles on cylindrical manifolds where the compactified dimension can be collapsed or inflated at will so as to connect continuously the two extreme cases. Second, we shall focus on a system that is fully under analytical control, that is the Kraichnan ensemble of velocities rather than Navier–Stokes turbulence.

To study the turbulent transport of a passive scalar, Kraichnan introduced a Gaussian ensemble of decorrelated-in-time velocity fields [19]. A compressible generalization of the Kraichnan ensemble in the  $d$ -dimensional Euclidean space has been investigated by Gawędzki and Vergassola under the assumption of statistical isotropy [15].<sup>1</sup> In this model, the dynamics of fluid particles depends on three physical quantities: the space dimension, the degree of compressibility, and the (spatial) Hölder exponent of the velocity. The Hölder exponent  $\xi/2$  is greater than zero and less than one. This property mimics the behavior of a turbulent velocity field, whose realizations are typically non-Lipschitz in the limit of infinite Reynolds number. For any given  $d < 4$  and  $0 < \xi < 2$ , Gawędzki and Vergassola have identified a critical degree of compressibility separating two different phases of the Lagrangian dynamics. Below the critical value (incompressible or weakly compressible velocity fields), fluid particles separate superdiffusively. The probability distribution of fluid-particle separations does not have a stationary limit in this regime. Above the threshold (strongly compressible fields), Lagrangian trajectories tend to collapse to zero distance, and the distribution of the separations degenerates into a Dirac delta function. For  $d \geq 4$ , the former regime

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<sup>1</sup> The smooth limit of this model had been previously considered in ref. [3].

is the only possible one and the phase transition does not occur.<sup>2</sup> The above results have been subsequently elaborated by Le Jan and Raimond in the context of non-Lipschitz stochastic differential equations [21,22].

Here we consider a generalization of the Kraichnan ensemble on a cylindrical surface. A  $d$ -dimensional cylindrical surface can be constructed by taking  $\mathbb{R}^d$  and compactifying  $d - d'$  dimensions. The radius of the cylinder is the size of the compactified dimensions. When the radius tends to infinity we recover  $\mathbb{R}^d$ ; when it tends to zero we obtain  $\mathbb{R}^{d'}$ . Thus, varying the radius of the cylinder produces a smooth transformation from dimension  $d$  to dimension  $d'$ .

We define a zero-mean Gaussian velocity field on a cylindrical surface by imposing the form of its covariance. We require that the covariance of the field tends to the one of the isotropic  $d$ -dimensional Kraichnan ensemble as the radius of the cylinder tends to infinity and to the one of the isotropic  $d'$ -dimensional Kraichnan ensemble as the radius vanishes. The degree of compressibility is such that the velocity is weakly compressible in the limit of infinite radius and strongly compressible in the opposite limit. It is therefore possible to gradually move from one regime to the other by varying the radius of the cylinder.

As we shall see, if in the limit of infinite radius the Hölder exponent is equal to  $\xi$ , then in the limit of vanishing radius it is equal to  $\xi + (d - d')$ . Hence, if attention is restricted to non-smooth velocities, the model under consideration is meaningful only when a single dimension is compactified ( $d' = d - 1$ ). For the sake of simplicity, we shall conduct the analysis in two dimensions, where the two extreme cases are the two-dimensional plane and the straight line. We shall show that the dynamics of fluid particles results from two opposite effects. At small separations, Lagrangian trajectories exhibit a superdiffusive dynamics owing to the weakly compressible nature of the small-scale velocity. At large separations, fluid particles experience the trapping effect of a strongly compressible field. Consequently, the probability distribution of the two-particle separation tends to an invariant measure. This behavior is to be contrasted with the one observed in the two-dimensional isotropic case with the same Hölder exponent and degree of compressibility.

In the present context, the separation vector between two fluid particles is a stochastic process solving an Itô stochastic differential equation with non-Lipschitz diffusion coefficient. To guarantee the existence and the uniqueness in law of the solution, we shall add pure diffusion to the velocity field. By considering an appropriate Lyapunov function, we shall demonstrate that there exists an invariant measure for the fluid-particle separation. The invariant measure is unique, ergodic, and non-degenerate as a consequence of the irreducibility and the strong Feller property of the process.

The paper is divided as follows. Section 2 describes a generalization of the Kraichnan model on a  $d$ -dimensional cylindrical surface. The two-dimensional case is studied in detail in section 3. Sections 4 and 5 contain the results on the fluid-particle separation and its invariant measure. The limit of vanishing diffusivity, the effect of a viscous regularization of the velocity field, and the role of the Prandtl number are discussed in section 6.

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<sup>2</sup> For  $d = 4$ , the collapsed phase can exist only for smooth velocity fields ( $\xi = 2$ ).

## 2 Kraichnan model on a $d$ -dimensional cylindrical surface

We consider the dynamics of fluid particles in a turbulent flow on a  $d$ -dimensional cylindrical surface  $\mathcal{S}$ . The velocity field is a family of white noises taking their values in the space of vector fields on  $\mathcal{S}$ . Specifically,  $\mathbf{v}(t, \mathbf{x})$  is a Gaussian stochastic process with zero mean and covariance

$$\mathbb{E}(v_\alpha(t, \mathbf{x})v_\beta(s, \mathbf{y})) = D_{\alpha\beta}(\mathbf{x} - \mathbf{y})\delta(t - s), \quad (1)$$

where  $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{d'} \times [-\pi L, \pi L)^{d-d'} \subset \mathbb{R}^d$  ( $d > d'$ ) and  $L$  is the radius of the cylinder. The velocity is by definition statistically homogeneous in space, stationary in time, and invariant under time reversal. Moreover, we assume periodicity in the  $d - d'$  “radial” coordinates.

It is convenient to write the spatial covariance  $D_{\alpha\beta}(\mathbf{r})$  in terms of its Fourier-space representation:

$$D_{\alpha\beta}(\mathbf{r}) = \frac{1}{(2\pi)^{d'}(2\pi L)^{d-d'}} \sum_{\mathbf{k}'' \in \frac{1}{L}\mathbb{Z}^{d-d'}} e^{i\mathbf{k}'' \cdot \mathbf{r}''} \int_{\mathbb{R}^{d'}} d\mathbf{k}' e^{i\mathbf{k}' \cdot \mathbf{r}'} F_{\alpha\beta}(\mathbf{k})$$

with  $\mathbf{k} = (\mathbf{k}', \mathbf{k}'') \in \mathbb{R}^{d'} \times \frac{1}{L}\mathbb{Z}^{d-d'}$ ,  $\mathbf{r} = (\mathbf{r}', \mathbf{r}'') \in \mathbb{R}^{d'} \times [-\pi L, \pi L)^{d-d'}$ , and  $\alpha, \beta = 1, \dots, d$ . The presence of a series in the  $\mathbf{k}''$ -coordinates accounts for the periodicity of the velocity field in the  $\mathbf{r}''$ -coordinates.

We adopt the following form for the spectral tensor:

$$F_{\alpha\beta}(\mathbf{k}) = \frac{A_{\alpha\beta}(\mathbf{k}; \wp)}{(\|\mathbf{k}\|^2 + \ell^{-2})^{\frac{d+\xi}{2}}} \quad (2)$$

with  $\ell \in \mathbb{R}_+$ ,  $\xi \in [0, 2]$ ,  $\wp \in [0, 1]$ , and

$$A_{\alpha\beta}(\mathbf{k}; \wp) = (1 - \wp)\delta_{\alpha\beta} + (\wp d - 1) \frac{k_\alpha k_\beta}{\|\mathbf{k}\|^2}.$$

As we shall see in the latter part of this section,  $F_{\alpha\beta}(\mathbf{k})$  has been chosen in such a way that, in the limits  $L \rightarrow 0$  and  $L \rightarrow \infty$ ,  $D_{\alpha\beta}(\mathbf{r})$  tends to the covariance of an isotropic random field.<sup>3</sup>

The spectral tensor is real, symmetric ( $F_{\alpha\beta}(\mathbf{k}) = F_{\beta\alpha}(\mathbf{k})$ ) and non-negative definite  $\forall \mathbf{k} \in \mathbb{R}^{d'} \times \frac{1}{L}\mathbb{Z}^{d-d'}$ , i.e.,

$$\sum_{1 \leq \alpha, \beta \leq d} F_{\alpha\beta}(\mathbf{k}) u_\alpha u_\beta \geq 0 \quad \forall (u_1, \dots, u_d) \in \mathbb{R}^d,$$

as can be checked using the Cauchy–Schwartz inequality. These properties guarantee that  $D_{\alpha\beta}(\mathbf{r})$  is the spatial covariance of a homogeneous random field (e.g., ref. [25], p. 20). Moreover,  $F_{\alpha\beta}(\mathbf{k})$  is an even function of  $\mathbf{k}$ , and therefore the velocity is statistically invariant under parity:  $D_{\alpha\beta}(-\mathbf{r}) = D_{\alpha\beta}(\mathbf{r})$ .

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<sup>3</sup> The spectral tensor could in principle be multiplied by a positive coefficient determining the intensity of the velocity fluctuations. For the sake of simplicity, we set that coefficient to one.

As a consequence of statistical homogeneity and parity invariance, the covariance of velocity differences can be expressed in terms of  $D_{\alpha\beta}(\mathbf{r})$ :

$$\mathbb{E}([v_\alpha(t, \mathbf{x} + \mathbf{r}) - v_\alpha(t, \mathbf{x})][v_\beta(s, \mathbf{x} + \mathbf{r}) - v_\beta(s, \mathbf{x})]) = 2d_{\alpha\beta}(\mathbf{r})\delta(t - s) \quad (3)$$

with  $d_{\alpha\beta}(\mathbf{r}) = D_{\alpha\beta}(0) - D_{\alpha\beta}(\mathbf{r})$  [25].

The meaning of the parameters  $\wp$ ,  $\ell$ , and  $\xi$  may be understood by considering the limit of  $D_{\alpha\beta}(\mathbf{r})$  for  $L \rightarrow \infty$  and for  $L \rightarrow 0$ .

The limit  $L \rightarrow \infty$  (and  $1/L \rightarrow d\mathbf{k}''$ ) yields:

$$\begin{aligned} \lim_{L \rightarrow \infty} D_{\alpha\beta}(\mathbf{r}) &= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^{d-d'}} d\mathbf{k}'' \int_{\mathbb{R}^{d'}} d\mathbf{k}' \frac{e^{i\mathbf{k}' \cdot \mathbf{r}' + i\mathbf{k}'' \cdot \mathbf{r}''} A_{\alpha\beta}(\mathbf{k}; \wp)}{(\|\mathbf{k}'\|^2 + \|\mathbf{k}''\|^2 + \ell^{-2})^{\frac{d+\xi}{2}}} \\ &= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} d\mathbf{k} \frac{e^{i\mathbf{k} \cdot \mathbf{r}} A_{\alpha\beta}(\mathbf{k}; \wp)}{(\|\mathbf{k}\|^2 + \ell^{-2})^{\frac{d+\xi}{2}}}. \end{aligned} \quad (4)$$

In this limit,  $D_{\alpha\beta}(\mathbf{r})$  tends to the spatial covariance of a  $d$ -dimensional isotropic field with correlation length  $\ell$  and degree of compressibility  $\wp$  [15, 11]. The parameter  $\xi/2$  represents the inertial-range Hölder exponent of the velocity:  $\sum_{\alpha=1}^d d_{\alpha\alpha}(\mathbf{r}) = O(\|\mathbf{r}\|^\xi)$  as  $\|\mathbf{r}/\ell\| \rightarrow 0$ . For  $\xi = 0$  the velocity field is purely diffusive; for  $\xi = 2$  it is spatially smooth, and its spatial regularity decreases with decreasing  $\xi$ . In particular, the Kolmogorov scaling is obtained for  $\xi = 4/3$ , for the time integral of (3) must be proportional to  $\|\mathbf{r}\|^{4/3}$  in Kolmogorov's phenomenology [13].<sup>4</sup>

It is worth noting that for a finite  $L$  equation (4) describes the velocity covariance at space separations much smaller than  $L$ .

In the second limit,  $L \rightarrow 0$ , we obtain<sup>5</sup>

$$\lim_{L \rightarrow 0} D_{\alpha\beta}(\mathbf{r}) = \delta(\mathbf{r}'') \frac{\mathcal{K}}{(2\pi)^{d'}} \int_{\mathbb{R}^{d'}} d\mathbf{k}' \frac{e^{i\mathbf{k}' \cdot \mathbf{r}'} A_{\alpha\beta}(\mathbf{k}'; \wp')}{(\|\mathbf{k}'\|^2 + \ell^{-2})^{\frac{d'+\xi'}{2}}}$$

with  $\xi' = \xi + (d - d')$  and

$$\wp' = \frac{\wp(d-1)}{\wp(d-d') + d' - 1}, \quad \mathcal{K} = 1 + \frac{\wp(d-d')}{d' - 1} \quad \text{if } d' > 1,$$

$$A_{11}(k'; \wp') = 1, \quad \mathcal{K} = \wp(d-1) \quad \text{if } d' = 1 \text{ and } \wp > 0.$$

<sup>4</sup> The same conclusion can be reached rigorously by defining the Kraichnan ensemble as the limit of an Ornstein-Uhlenbeck process for vanishing correlation time [10].

<sup>5</sup> This can be shown by multiplying  $D_{\alpha\beta}(\mathbf{r})$  by a function  $f(\mathbf{r}'')$ , integrating over  $\mathbf{r}'' \in [-\pi L, \pi L)$ , taking the limit  $L \rightarrow 0$ , and noting that only the term corresponding to  $\mathbf{k}'' = 0$  has a non-zero limit equal to

$$f(0) \frac{\mathcal{K}}{(2\pi)^{d'}} \int_{\mathbb{R}^{d'}} d\mathbf{k}' \frac{e^{i\mathbf{k}' \cdot \mathbf{r}'} A_{\alpha\beta}(\mathbf{k}'; \wp')}{(\|\mathbf{k}'\|^2 + \ell^{-2})^{\frac{d'+\xi'}{2}}}.$$

We thus recover the covariance of a  $d'$ -dimensional isotropic velocity field with Hölder exponent  $\xi'/2$ , correlation length  $\ell$ , and degree of compressibility  $\wp'$  [15, 11].

The exponents  $\xi$  and  $\xi'$  must satisfy the inequalities  $0 \leq \xi \leq 2$  and  $0 \leq \xi' \leq 2$ . Therefore, the limit  $L \rightarrow 0$  makes sense only in two cases:

- a)  $d' = d - 1$ ,  $\xi \in [0, 1]$ , and  $\xi' = \xi + 1 \in [1, 2]$ ;
- b)  $d' = d - 2$ ,  $\xi = 0$ , and  $\xi' = 2$ .

We are interested in the situation where the fluid particles disperse when  $L \rightarrow \infty$  ( $d$ -dimensional isotropic flow) and collapse when  $L \rightarrow 0$  ( $d'$ -dimensional isotropic flow). Moreover, we focus on spatially rough velocity fields leaving aside the cases  $\xi = 0$  and  $\xi' = 2$ . This situation can be realized only in case a), for  $\xi \in (0, 1)$ , and under the conditions [15]:

$$\wp < \frac{d}{\xi^2} \quad \text{and} \quad \wp' \geq \frac{d'}{\xi'^2}.$$

The first inequality is actually satisfied for all  $d$  and  $\xi$  given that  $\xi \in (0, 1)$  and  $\wp \in [0, 1]$ . The second inequality can be rewritten in terms of  $\wp$  as follows:

$$\wp \geq \frac{d-2}{\xi(\xi+2)}. \quad (5)$$

The restriction  $0 \leq \wp \leq 1$  and inequality (5) imply the additional condition  $d < 5$ .

In the remainder of the paper, we shall investigate the statistics of fluid-particle separations on a two-dimensional cylindrical surface ( $d = 2$ ).

### 3 Two-dimensional cylindrical surface

For  $d = 2$ , case a) is the only realizable one, corresponding to  $d' = 1$ . Condition (5) reduces to  $\wp \geq 0$  independently of  $\xi$ .

The spatial covariance of the velocity field takes the form

$$D_{\alpha\beta}(\mathbf{r}) = \frac{1}{4\pi^2 L} \sum_{j=-\infty}^{\infty} e^{i\frac{j}{L}r_2} \int_{\mathbb{R}} dk_1 \frac{e^{ik_1 r_1} A_{\alpha\beta}((k_1, \frac{j}{L}); \wp)}{[k_1^2 + (\frac{j}{L})^2 + \frac{1}{\ell^2}]^{\frac{2+\xi}{2}}} \quad (6)$$

with  $\mathbf{r} = (r_1, r_2) \in \Omega = \mathbb{R} \times [-\pi L, \pi L]$ . In eq. (6) we have written  $\mathbf{k} = (k_1, k_2)$  with  $k_2 = j/L$  and  $j \in \mathbb{Z}$  to make the dependence on  $L$  explicit. We shall keep this notation in the remainder of the paper.

We now restrict attention to space separations much smaller than  $\ell$ . Formally, this is equivalent to considering the limit  $\ell \rightarrow \infty$ . The spatial variance of the velocity field,  $D_{\alpha\beta}(\mathbf{0})$ , diverges as  $\ell$  tends to infinity (appendix A); this behavior reflects the divergence of the average kinetic energy of the fluid. Nevertheless,  $d_{\alpha\beta}(\mathbf{r})$  has a finite limit for all  $\mathbf{r}$ , and the statistics of velocity differences remains well defined.

The limit of  $d_{\alpha\beta}(\mathbf{r})$  for  $\ell \rightarrow \infty$  can be computed explicitly (appendix A). The correlation of the axial component is written:

$$\begin{aligned} \lim_{\ell \rightarrow \infty} d_{11}(\mathbf{r}) &= \frac{\wp \left| \Gamma \left( -\frac{1+\xi}{2} \right) \right|}{2^{3+\xi} \pi^{3/2} \Gamma \left( 1 + \frac{\xi}{2} \right) L} |r_1|^{1+\xi} \\ &+ \frac{L^\xi}{2\pi^{3/2} \Gamma \left( 2 + \frac{\xi}{2} \right)} \sum_{j=1}^{\infty} j^{-1-\xi} \left\{ \frac{1 + (1-\wp)\xi}{2} \Gamma \left( \frac{1+\xi}{2} \right) \right. \\ &- 2 \cos \left( \frac{j r_2}{L} \right) \left[ \wp \left( 1 + \frac{\xi}{2} \right) \left( \frac{j|r_1|}{2L} \right)^{\frac{1+\xi}{2}} K_{\frac{1+\xi}{2}} \left( \frac{j|r_1|}{L} \right) \right. \\ &\quad \left. \left. + (1-2\wp) \left( \frac{j|r_1|}{2L} \right)^{\frac{3+\xi}{2}} K_{\frac{3+\xi}{2}} \left( \frac{j|r_1|}{L} \right) \right] \right\}, \quad (7) \end{aligned}$$

where  $K_\nu(z)$  denotes the modified Bessel function of the second kind of order  $\nu$  and argument  $z$ . The correlation of the radial component has the form:

$$\begin{aligned} \lim_{\ell \rightarrow \infty} d_{22}(\mathbf{r}) &= \frac{(1-\wp) \left| \Gamma \left( -\frac{1+\xi}{2} \right) \right|}{2^{3+\xi} \pi^{3/2} \Gamma \left( 1 + \frac{\xi}{2} \right) L} |r_1|^{1+\xi} \\ &+ \frac{L^\xi}{2\pi^{3/2} \Gamma \left( 2 + \frac{\xi}{2} \right)} \sum_{j=1}^{\infty} j^{-1-\xi} \left\{ \frac{1 + \wp\xi}{2} \Gamma \left( \frac{1+\xi}{2} \right) \right. \\ &- 2 \cos \left( \frac{j r_2}{L} \right) \left[ (1-\wp) \left( 1 + \frac{\xi}{2} \right) \left( \frac{j|r_1|}{2L} \right)^{\frac{1+\xi}{2}} K_{\frac{1+\xi}{2}} \left( \frac{j|r_1|}{L} \right) \right. \\ &\quad \left. \left. + (2\wp-1) \left( \frac{j|r_1|}{2L} \right)^{\frac{3+\xi}{2}} K_{\frac{3+\xi}{2}} \left( \frac{j|r_1|}{L} \right) \right] \right\}. \quad (8) \end{aligned}$$

Finally, the mixed correlations can be written as follows:

$$\begin{aligned} \lim_{\ell \rightarrow \infty} d_{12}(\mathbf{r}) &= \lim_{\ell \rightarrow \infty} d_{21}(\mathbf{r}) \\ &= \frac{(2\wp-1)L^\xi}{2\pi^{3/2} \Gamma \left( 2 + \frac{\xi}{2} \right)} \sum_{j=1}^{\infty} j^{-1-\xi} \left( \frac{j r_1}{L} \right) \sin \left( \frac{j r_2}{L} \right) \left( \frac{j|r_1|}{2L} \right)^{\frac{1+\xi}{2}} K_{\frac{1+\xi}{2}} \left( \frac{j|r_1|}{L} \right). \end{aligned} \quad (9)$$

The limit  $\ell \rightarrow \infty$  will be hereafter understood.

### 3.1 Large-scale form of the covariance of velocity differences

To understand the nature of the random velocity field, it is useful to consider the covariance of velocity differences at space separations much greater than the radius of the cylinder.

The series in eqs. (7)-(9) converge uniformly (appendix A). For  $|r_1|/L \rightarrow \infty$ , it is therefore possible to deduce the asymptotic expansion of  $d_{\alpha\beta}(\mathbf{r})$  from the limiting behavior of the single terms of the series. The asymptotic expansion of  $K_\nu(z)$  for  $z \rightarrow \infty$  is (e.g., ref. [9], formula II 7.13(7))

$$K_\nu(z) \sim \sqrt{\frac{\pi}{2z}} e^{-z} \quad (|\arg z| < 3\pi/2).$$

Thus, the  $r_2$ -dependent contributions to  $d_{\alpha\beta}(\mathbf{r})$  decay exponentially fast with increasing space separation. The remaining contributions give

$$d_{11}(\mathbf{r}) \sim \mathfrak{D}_1 |r_1|^{1+\xi} + \varkappa_1 \quad \text{as } \frac{|r_1|}{L} \rightarrow \infty \quad (10)$$

with

$$\mathfrak{D}_1 = \frac{\wp \left| \Gamma \left( -\frac{1+\xi}{2} \right) \right|}{2^{3+\xi} \pi^{3/2} \Gamma \left( 1 + \frac{\xi}{2} \right) L}$$

and

$$\varkappa_1 = \frac{[1 + (1 - \wp)\xi] L^\xi \Gamma \left( \frac{1+\xi}{2} \right)}{2\pi^{3/2} (2 + \xi) \Gamma \left( 1 + \frac{\xi}{2} \right)} \zeta(1 + \xi).$$

In the latter equation

$$\zeta(s) = \sum_{j=1}^{\infty} \frac{1}{j^s} \quad (s > 1)$$

is the Riemann Zeta function. Likewise, we have

$$d_{22}(\mathbf{r}) \sim \mathfrak{D}_2 |r_1|^{1+\xi} + \varkappa_2 \quad \text{as } \frac{|r_1|}{L} \rightarrow \infty \quad (11)$$

with

$$\mathfrak{D}_2 = \frac{1 - \wp}{\wp} \mathfrak{D}_1, \quad \varkappa_2 = \frac{1 + \wp\xi}{1 + (1 - \wp)\xi} \varkappa_1.$$

Finally, the off-diagonal terms vanish at large space separations:

$$\lim_{|r_1|/L \rightarrow \infty} d_{12}(\mathbf{r}) = \lim_{|r_1|/L \rightarrow \infty} d_{21}(\mathbf{r}) = 0. \quad (12)$$

The above asymptotic expressions show that, at separations much greater than the radius of the cylinder, the velocity difference may be regarded as the superposition of two independent one-dimensional random fields. One field is directed along the axial direction; the other one is directed along the radial direction. Both the fields depend only on  $r_1$ . In particular, the axial field is a one-dimensional Kraichnan velocity field with Hölder exponent  $1 + \xi$  like the one considered in ref. [30].

At large separations, the small-scale dynamics manifests itself through an effective diffusivity represented by the constants  $\varkappa_1$  and  $\varkappa_2$ .



#### 4 Fluid-particle dynamics

In the present context, the separation between two fluid particles can be regarded as a stochastic process on  $\Omega$  with diffusion coefficient  $d_{\alpha\beta}(\mathbf{r})$  (and drift coefficient equal to zero). To ensure the (weak) existence and uniqueness of the trajectories of the process, we add diffusion to the velocity field and replace  $d_{\alpha\beta}(\mathbf{r})$  by

$$d_{\alpha\beta}^{\kappa}(\mathbf{r}) := d_{\alpha\beta}(\mathbf{r}) + 2\kappa\delta_{\alpha\beta}, \quad \kappa > 0.$$

The additional term can model the action of molecular diffusion on fluid particles as, e.g., in ref. [15]. The constant  $\kappa$  will be referred to as diffusivity.

The separation vector between two fluid particles will be denoted by  $\mathbf{R}$ . According to the above remark,  $\mathbf{R}$  satisfies the Itô stochastic differential equation<sup>6</sup>

$$d\mathbf{R}(t) = \sqrt{2}\sigma(\mathbf{R}(t))d\mathbf{B}(t), \quad \mathbf{R}(0) = \mathbf{r} \in \Omega, \quad (13)$$

where  $\mathbf{B}$  is Brownian motion on  $\Omega$  and  $\sigma$  is defined through the Cholesky decomposition of the matrix  $d^{\kappa}$ :

$$\sigma\sigma^{\text{T}} = d^{\kappa}$$

with

$$\sigma(\mathbf{r}) = \begin{pmatrix} \sqrt{d_{11}^{\kappa}(\mathbf{r})} & 0 \\ \frac{d_{12}^{\kappa}(\mathbf{r})}{\sqrt{d_{11}^{\kappa}(\mathbf{r})}} & \sqrt{d_{22}^{\kappa}(\mathbf{r}) - \frac{[d_{12}^{\kappa}(\mathbf{r})]^2}{d_{11}^{\kappa}(\mathbf{r})}} \end{pmatrix}, \quad \mathbf{r} \in \Omega.$$

Although the diffusion matrix  $d^{\kappa}$  is not Lipschitz continuous nor bounded, the existence and uniqueness of the solution of eq. (13) can be proved using Stroock's and Varadhan's theory of martingale problems [28]. To directly exploit this theory, we shall first consider the periodic extension of eq. (13) on  $\mathbb{R}^2$ , and then project the resulting process on  $\Omega$ . We therefore introduce the projection  $\mathbf{p} : \mathbb{R}^2 \rightarrow \Omega$  with

$$\mathbf{p}(\mathbf{r}) = \left( r_1, -\pi L + r_2 - 2\pi L \left\lfloor \frac{r_2}{2\pi L} \right\rfloor \right),$$

and define  $\tilde{\sigma} := \sigma \circ \mathbf{p}$ . Likewise we denote  $\tilde{d}^{\kappa} := d^{\kappa} \circ \mathbf{p} = \tilde{\sigma}\tilde{\sigma}^{\text{T}}$ .

Before proceeding further, it is convenient to define some notation. The spaces of bounded measurable and bounded continuous functions on  $\Omega$  will be denoted by  $\mathcal{B}_b(\Omega)$  and  $\mathcal{C}_b(\Omega)$ , respectively. The set  $\mathcal{C}^2(\Omega)$  will be the space of functions having two continuous derivatives. Analogous definitions will apply to functions defined on  $\mathbb{R}^2$ .

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<sup>6</sup> If  $\mathbf{X}$  and  $\mathbf{Y}$  denote the positions of two fluid particles, the separation vector between the two particles is defined as  $\mathbf{R} := \mathbf{Y} - \mathbf{X}$ . The common physical notation for the evolution equation for  $\mathbf{R}$  would be

$$\frac{d\mathbf{R}}{dt} = \delta_{\mathbf{R}}\mathbf{v}(t) + 2\sqrt{\kappa}\boldsymbol{\xi}(t)$$

where  $\boldsymbol{\xi}$  is white noise and the statistics of  $\delta_{\mathbf{R}}\mathbf{v}(t) := \mathbf{v}(t, \mathbf{Y}(t)) - \mathbf{v}(t, \mathbf{X}(t))$  is defined by eq. (3)

**Proposition 1** *The Itô stochastic differential equation on  $\mathbb{R}^2$ :*

$$d\tilde{\mathbf{R}}(t) = \sqrt{2} \tilde{\sigma}(\tilde{\mathbf{R}}(t)) d\tilde{\mathbf{B}}(t), \quad \tilde{\mathbf{R}}(0) = \mathbf{r} \in \mathbb{R}^2, \quad (14)$$

where  $\tilde{\mathbf{B}}$  is the standard Brownian motion on  $\mathbb{R}^2$ , has a unique (in law) weak solution. In particular, the solution is a continuous Markov process.

For  $\mathbf{r} \in \mathbb{R}^2$  and  $U \subseteq \mathbb{R}^2$  measurable, let

$$\tilde{P}(0, \mathbf{r}; s, U) := \mathbb{P} \left( \tilde{\mathbf{R}}(s) \in U \text{ if } \tilde{\mathbf{R}}(0) = \mathbf{r} \right)$$

be the transition probability distribution of  $\tilde{\mathbf{R}}$ , and let  $(\tilde{T}_t)_{t \geq 0}$  be the associate transition semigroup:

$$\tilde{T}_t f(\mathbf{r}) := \int_{\mathbb{R}^2} f(\boldsymbol{\rho}) \tilde{P}(0, \mathbf{r}; t, d\boldsymbol{\rho})$$

with  $f \in \mathcal{B}_b(\mathbb{R}^2)$ . Then, the semigroup  $(\tilde{T}_t)_{t \geq 0}$  has the strong Feller property, i.e.  $\tilde{T}_t(\mathcal{B}_b(\mathbb{R}^2)) \subset \mathcal{C}_b(\mathbb{R}^2)$  for all  $t > 0$ .

*Proof* The diffusion matrix has the following properties:

1.  $\tilde{d}^\kappa$  is continuous;
2.  $\tilde{d}^\kappa(\mathbf{r})$  is symmetric and strictly positive definite for all  $\mathbf{r} \in \mathbb{R}^2$ . The diffusivity  $\kappa$  is indeed assumed to be strictly positive, and the spatial covariance of velocity differences must be symmetric and uniformly non-negative definite for all  $\mathbf{r}$  (ref. [25], p. 97), i.e.,

$$\sum_{1 \leq \alpha, \beta \leq 2} d_{\alpha\beta}(\mathbf{r}) u_\alpha u_\beta \geq 0 \quad \forall (u_1, u_2) \text{ and } \mathbf{r} \in \mathbb{R}^2;$$

3. there exists a positive constant  $C_1$  such that for all  $\alpha, \beta$ , and  $\mathbf{r}$

$$|\tilde{d}_{\alpha\beta}^\kappa(\mathbf{r})| \leq C_1(1 + \|\mathbf{r}\|^2). \quad (15)$$

This property is a consequence of the asymptotic behaviors (10) to (12) and of the fact that  $\tilde{d}^\kappa(\mathbf{r})$  is bounded at the origin and continuous on  $\mathbb{R}^2$ .

Under the above conditions, Stroock's and Varadhan's uniqueness theorem apply to the martingale problem for  $\tilde{d}^\kappa$  [28]. Then, the proposition follows from the equivalence between the well-posedness of martingale problems and the existence and uniqueness in law of weak solutions of stochastic differential equations (ref. [27], pp. 159 and 170).  $\square$

The process  $\mathbf{R}$  can be regarded as the projection of  $\tilde{\mathbf{R}}$  on  $\Omega$ :  $\mathbf{R}(t) = \mathbf{p}(\tilde{\mathbf{R}}(t))$ . The properties of  $\mathbf{R}$  can then be deduced from those of  $\tilde{\mathbf{R}}$ .

**Corollary 1** *Equation (13) has a unique (in law) weak solution. The transition semigroup  $(T_t)_{t \geq 0}$  associated with  $\mathbf{R}$  has the strong Feller property:  $T_t(\mathcal{B}_b(\Omega)) \subset \mathcal{C}_b(\Omega)$  for all  $t > 0$ .*

*Proof* Given an initial condition  $\mathbf{r} \in \Omega$ , a weak solution of eq. (13) can be constructed by taking a solution of eq. (14) with the same initial condition and projecting it on  $\Omega$ .

The key observation to prove uniqueness in law is that any solution of eq. (13) on  $\Omega$  can be uniquely mapped into a continuous solution of eq. (14) on  $\mathbb{R}^2$ . Then, uniqueness in law in  $\mathbb{R}^2$  guarantees that also the solution on  $\Omega$  is unique in law.

Finally,  $(\tilde{T}_t)_{t \geq 0}$  has the strong Feller property and the projection  $\mathbf{p}$  is locally invertible (with continuous inverse). Hence,  $(T_t)_{t \geq 0}$  has the strong Feller property.  $\square$

## 5 Invariant measure of fluid-particle separations

An invariant measure for  $\mathbf{R}$  can be constructed by adapting to the case under examination the procedure described in ref. [26]. Clearly, an invariant measure may exist only if the trajectories of the stochastic process do not “escape to infinity”. To control the behavior of the first component of  $\mathbf{R}$ , which is not bounded, we therefore introduce the Lyapunov function  $V : \Omega \rightarrow \mathbb{R}_+$ :

$$V(\mathbf{r}) = \begin{cases} \frac{h(h+1)c^4 + 2(1-h^2)c^2r_1^2 - h(1-h)r_1^4}{4(1-h)c^{2(h+1)}} & \text{if } |r_1| \leq c, \\ \frac{1}{2(1-h)} |r_1|^{2(1-h)} & \text{if } |r_1| > c, \end{cases}$$

where  $c > 0$  and  $1 > h > 0$ . The function  $V$  is twice continuously differentiable and has the asymptotic behavior needed for the proof.

**Lemma 1** *If  $(1+\xi)/2 > h > 1/2$  and  $\mathcal{A}$  denotes the infinitesimal generator of  $(T_t)_{t \geq 0}$ , then the Lyapunov function has the following properties for all  $\xi \in (0, 1)$ :*

1.  $\lim_{\|\mathbf{r}\| \rightarrow \infty} \mathcal{A}V(\mathbf{r}) = -\infty$ ;
2. *there exists  $m \in \mathbb{R}$  such that  $\mathcal{A}V(\mathbf{r}) \leq m$  for all  $\mathbf{r} \in \Omega$ ;*
3.  $T_t V(\mathbf{r}) = V(\mathbf{r}) + \int_0^t T_s \mathcal{A}V(\mathbf{r}) ds$ .

*Proof* For  $f \in \mathcal{C}^2(\Omega)$ , the infinitesimal generator of  $(T_t)_{t \geq 0}$  has the form

$$\mathcal{A}f(\mathbf{r}) = \text{tr}[\sigma(\mathbf{r})\sigma^T(\mathbf{r})D^2f(\mathbf{r})],$$

where  $D^2f$  denotes the Hessian of the function  $f$ . The action of the generator  $\mathcal{A}$  on  $V(\mathbf{r})$  is written:

$$\mathcal{A}V(\mathbf{r}) = \sigma_{11}^2(\mathbf{r}) \frac{\partial^2 V}{\partial r_1^2}.$$

From eq. (10) we obtain

$$\mathcal{A}V(\mathbf{r}) \sim -(2h-1)\mathfrak{D}_1|r_1|^{1+\xi-2h} \quad \text{as } \|\mathbf{r}\| \rightarrow \infty.$$

If  $(1+\xi)/2 > h > 1/2$ , we have  $\lim_{\|\mathbf{r}\| \rightarrow \infty} \mathcal{A}V(\mathbf{r}) = -\infty$  for all  $\xi \in (0, 1)$ . It is worth noting that this latter result relies on the fact that  $d_{11}(\mathbf{r}) = O(|r_1|^{1+\xi})$  as  $\|\mathbf{r}\| \rightarrow \infty$  with  $1 + \xi > 1$ .

Property 2 is a consequence of the continuity of  $\mathcal{A}V$  and of property 1.

Finally, the transition semigroup satisfies

$$T_t f(\mathbf{r}) = f(\mathbf{r}) + \int_0^t T_s \mathcal{A}f(\mathbf{r}) ds$$

for any  $f \in \mathcal{C}_b^2(\Omega)$ . The same property holds true for the function  $V$ , as can be shown using an approximation procedure similar to the one described in ref. [26], pp. 167–168. The details are given in appendix B.  $\square$

We now make use of the properties of the Lyapunov function to obtain the following result.

**Proposition 2** *There exists an invariant measure  $\mu$  for the stochastic process  $\mathbf{R}$ , i.e.*

$$\int_{\Omega} (T_t f)(\mathbf{r}) \mu(d\mathbf{r}) = \int_{\Omega} f(\mathbf{r}) \mu(d\mathbf{r}) \quad (16)$$

for all  $f \in \mathcal{C}_b(\Omega)$  and for all  $t > 0$ .

*Proof* Given  $\mathbf{r}_0 \in \Omega$ , we have

$$\begin{aligned} \frac{1}{t} \int_0^t T_s (m - \mathcal{A}V)(\mathbf{r}_0) ds &= m - \frac{1}{t} \int_0^t T_s \mathcal{A}V(\mathbf{r}_0) ds \\ &= m + \frac{V(\mathbf{r}_0) - T_t V(\mathbf{r}_0)}{t} \leq m + \frac{V(\mathbf{r}_0)}{t}. \end{aligned}$$

Hence

$$\sup_{t \geq 1} \left[ \frac{1}{t} \int_0^t T_s (m - \mathcal{A}V)(\mathbf{r}_0) ds \right] < \infty. \quad (17)$$

We now introduce the family of “average” measures  $(\mu_t)_{t \geq 1}$  on  $\Omega$  defined, for any  $f \in \mathcal{C}_b(\Omega)$ , as

$$\int_{\Omega} f(\mathbf{r}) \mu_t(d\mathbf{r}) = \frac{1}{t} \int_0^t T_s f(\mathbf{r}_0) ds.$$

We show that  $(\mu_t)_{t \geq 1}$  is uniformly tight. For a given  $N > 0$  we define

$$E_N = \{\mathbf{r} \in \Omega : m - \mathcal{A}V(\mathbf{r}) \leq N\}.$$

The set  $E_N$  is compact: it is closed since it is the preimage of a closed subset of  $\Omega$ , and must be bounded since  $\mathcal{A}V(\mathbf{r}) \rightarrow -\infty$  as  $\|\mathbf{r}\| \rightarrow \infty$ . As a consequence of Markov’s inequality, the measure of the complement of  $E_N$  satisfies:

$$\mu_t(E_N^c) \leq \frac{1}{N} \int_{\Omega} [m - \mathcal{A}V(\mathbf{r})] \mu_t(d\mathbf{r}) = \frac{1}{Nt} \int_0^t T_s (m - \mathcal{A}V)(\mathbf{r}_0) ds.$$

Using eq. (17) we conclude that for all  $\epsilon > 0$  there exists  $E_N \subset \Omega$  with

$$N = \frac{1}{\epsilon} \sup_{t \geq 1} \left[ \frac{1}{t} \int_0^t T_s(m - \mathcal{A}V)(\mathbf{r}_0) ds \right]$$

such that  $\mu_t(E_N^c) \leq \epsilon$  for all  $t \geq 1$ . The family  $(\mu_t)_{t \geq 1}$  is therefore uniformly tight. Then, there exists a measure  $\mu$  and a sequence  $(t_p)_{p \geq 0}$  with  $\lim_{p \rightarrow \infty} t_p = \infty$  such that  $\mu_{t_p}$  converges weakly to  $\mu$  as  $p \rightarrow \infty$ . This means that  $\int_{\Omega} f d\mu_{t_p} \rightarrow \int_{\Omega} f d\mu$  as  $p \rightarrow \infty$  for all  $f \in \mathcal{C}_b(\Omega)$  (e.g., ref. [6], theorem 11.5.4, p. 404).

We now show that  $\mu$  is invariant. For any  $f \in \mathcal{C}_b(\Omega)$ , for  $t > 0$ , and for all  $p$  such that  $t_p \geq t$ , we have:

$$\begin{aligned} & \left| \int_{\Omega} f(\mathbf{r}) \mu_{t_p}(d\mathbf{r}) - \int_{\Omega} T_t f(\mathbf{r}) \mu_{t_p}(d\mathbf{r}) \right| \\ &= \left| \frac{1}{t_p} \int_0^{t_p} T_s f(\mathbf{r}_0) ds - \frac{1}{t_p} \int_0^{t_p} T_s T_t f(\mathbf{r}_0) ds \right| \\ &= \left| \frac{1}{t_p} \int_0^{t_p} T_s f(\mathbf{r}_0) ds - \frac{1}{t_p} \int_t^{t+t_p} T_s f(\mathbf{r}_0) ds \right| \\ &= \left| \frac{1}{t_p} \int_0^t T_s f(\mathbf{r}_0) ds - \frac{1}{t_p} \int_{t_p}^{t+t_p} T_s f(\mathbf{r}_0) ds \right| \leq \frac{2t \|f\|_{\infty}}{t_p}. \end{aligned}$$

Hence, for all  $t > 0$ ,

$$\int_{\Omega} f(\mathbf{r}) \mu_{t_p}(d\mathbf{r}) - \int_{\Omega} T_t f(\mathbf{r}) \mu_{t_p}(d\mathbf{r}) \rightarrow 0 \quad \text{as } p \rightarrow \infty.$$

The semigroup  $(T_t)_{t \geq 0}$  satisfies  $T_t(\mathcal{C}_b(\Omega)) \subset \mathcal{C}_b(\Omega)$  since  $(T)_{t \geq 0}$  has the strong Feller property. By using the weak convergence of  $\mu_{t_p}$  to  $\mu$ , we can thus conclude that (16) holds for all  $f \in \mathcal{C}_b(\Omega)$  and for all  $t > 0$ . The measure  $\mu$  is therefore invariant for  $\mathbf{R}$ .  $\square$

To show that the invariant measure is actually unique, we need the following result stating that  $\mathbf{R}$  has no closed invariant set different from the whole space.

**Lemma 2** *The semigroup  $(T_t)_{t \geq 0}$  is irreducible, i.e. the transition probabilities of  $\mathbf{R}$ ,  $P(0, \mathbf{r}; t, U)$ , are strictly positive for all  $t > 0$ , for all  $\mathbf{r} \in \Omega$ , and for all non-empty open sets  $U \subseteq \Omega$ .*

*Proof* For all  $\mathbf{r} \in \mathbb{R}^2$  the linear transformation associated with  $\tilde{\sigma}(\mathbf{r})$  is invertible, and therefore maps  $\mathbb{R}^2$  into itself. Hence, the semigroup  $(\tilde{T}_t)_{t \geq 0}$  is irreducible (ref. [29], theorem 24, p. 66).

The transition probabilities of  $\mathbf{R}$  are connected to those of  $\tilde{\mathbf{R}}$  as follows:  $P(0, \mathbf{r}; t, U) = \tilde{P}(0, \mathbf{r}^*; t, U^*)$  where  $U^* = \mathbf{p}^{-1}(U)$  and  $\mathbf{r}^*$  is any point in  $\mathbf{p}^{-1}(\{\mathbf{r}\})$ . Therefore,  $(T_t)_{t \geq 0}$  is irreducible.  $\square$

We can now state the main result regarding the invariant measure of  $\mathbf{R}$ .

**Theorem 1** *There exists a unique invariant measure  $\mu$  for the stochastic process  $\mathbf{R}$ . The measure  $\mu$  is ergodic and equivalent to any transition probability  $P(0, \mathbf{r}; t, U)$  with  $\mathbf{r} \in \Omega$ ,  $t > 0$ , and  $U \subseteq \Omega$  measurable. Moreover,  $\mu$  is absolutely continuous with respect to the Lebesgue measure, and is therefore non degenerate (i.e., broad in  $\mathbf{r}$ ).*

*Proof* We have already proved that  $\mu$  is invariant. Its uniqueness, ergodicity, and equivalence to any transition probability follow from the fact that the transition semigroup associated with  $\mathbf{R}$  has the strong Feller property and is irreducible [7, 17] (see also ref. [5], chapter 4).

To prove the absolute continuity of  $\mu$  with respect to the Lebesgue measure, we introduce the family of transition probabilities

$$Q_\lambda(\mathbf{r}, U) = \lambda \int_0^\infty e^{-\lambda s} P(0, \mathbf{r}; s, U) ds$$

with  $\mathbf{r} \in \Omega$  and  $U \subseteq \Omega$  measurable, as well as the associate transition semigroup

$$\mathcal{T}_\lambda f(\mathbf{r}) = \int_\Omega f(\mathbf{y}) Q_\lambda(\mathbf{r}, d\mathbf{y}).$$

Likewise, we define an analogous family  $\tilde{Q}_\lambda(\mathbf{r}, U)$  for the process  $\tilde{\mathbf{R}}$ . The measure  $\mu$  is invariant also for  $(\mathcal{T}_\lambda)_{\lambda \geq 0}$ :

$$\int_\Omega \mu(d\mathbf{y}) \mathcal{T}_\lambda f(\mathbf{y}) = \lambda \int_0^\infty ds e^{-\lambda s} \int_\Omega \mu(d\mathbf{y}) T_s f(\mathbf{y}) = \int_\Omega \mu(d\mathbf{y}) f(\mathbf{y})$$

for any  $f \in \mathcal{B}_b(\Omega)$ , and hence

$$\mu(U) = \int_\Omega \mu(d\mathbf{y}) Q_\lambda(\mathbf{y}, U) \quad (18)$$

for any measurable set  $U \subseteq \Omega$ .

For all  $\mathbf{r} \in \mathbb{R}^2$ , the measure  $\tilde{Q}_\lambda(\mathbf{r}, \cdot)$  is absolutely continuous with respect to the Lebesgue measure (see ref. [29], theorem 10, p. 24). It follows that  $Q_\lambda(\mathbf{r}, \cdot)$  has the same property for all  $\mathbf{r} \in \Omega$  since  $Q_\lambda(\mathbf{r}, U) = \tilde{Q}_\lambda(\mathbf{r}^*, \mathbf{p}^{-1}(U))$  for a given  $\mathbf{r}^* \in \mathbf{p}^{-1}\{\mathbf{r}\}$ . From eq. (18),  $\mu$  is therefore absolutely continuous with respect to the Lebesgue measure.  $\square$

As a consequence of lemma 2 and theorem 1, the transition probability of  $\mathbf{R}$  has a positive density with respect to the Lebesgue measure:  $P(0, \mathbf{r}; t, d\boldsymbol{\rho}) = p(0, \mathbf{r}; t, \boldsymbol{\rho}) d\boldsymbol{\rho}$ . The probability density function is the (possibly weak) solution of:

$$\partial_t p = \mathcal{M}p, \quad (19)$$

where, for  $f \in \mathcal{C}^2(\Omega)$ ,

$$\mathcal{M}f(\boldsymbol{\rho}) = \sum_{1 \leq \alpha, \beta \leq 2} \partial_{\rho_\alpha} \partial_{\rho_\beta} d_{\alpha\beta}^\kappa(\boldsymbol{\rho}) f(\boldsymbol{\rho}).$$

## 6 Conclusions

We have studied the dynamics of fluid particles in a compressible turbulent velocity field on a cylinder. The model that we have introduced is a generalization of the isotropic Kraichnan ensemble. Although the parameters of the velocity have been set in such a way as to produce explosive separation of the fluid particles in the isotropic limit ( $L \rightarrow \infty$ ), on the cylinder the probability distribution of the separation tends to an invariant measure. This behavior is a result of the compressibility effects generated at large scales by the compactification of the “radial” dimension.

The diffusivity  $\kappa$  has been taken strictly positive to guarantee the existence of solutions to eq. (13). The addition of Brownian motion to Lagrangian trajectories influences the dynamics of fluid particles at small separations. Therefore, the presence of a nonzero diffusivity may be relevant for the non-degeneracy of the invariant measure, but should not affect the existence of the invariant measure itself, which rather depends on the large-scale form of the velocity field. The limit  $\kappa \rightarrow 0$  may be tackled by means of Wiener chaos decomposition methods [20, 21, 22, 23]. We conjecture that our results remain valid in that limit. Indeed, in the situation considered, the small-scale dynamics of fluid particles is the same as in the weakly compressible phase of the isotropic Kraichnan ensemble. In that regime, Lagrangian trajectories separate in time even for vanishing  $\kappa$  owing to the poor spatial regularity of the velocity [15]. Thus, the invariant measure should remain non-degenerate as  $\kappa \rightarrow 0$ .

The Reynolds number is infinite in our study since the viscosity of the fluid,  $\nu$ , has been set to zero from the beginning. For the same reason the Prandtl number  $Pr = \nu/\kappa$  is equal to zero. The viscosity can be taken into account by multiplying the spectral tensor (2) by the factor  $e^{-\eta^2 \|k\|^2}$ , where  $\eta \propto \nu^{3/4}$  plays the role of the viscous length of the flow [11]. This modification has a small-scale regularizing effect on the velocity field, which for any  $\eta > 0$  is locally Lipschitz continuous. Obviously, a positive  $\eta$  does not alter the proofs of the results shown in the paper.

The order of the limits  $\kappa \rightarrow 0$  and  $\eta \rightarrow 0$ , however, deserves a detailed discussion. Taking the limit  $\kappa \rightarrow 0$  before  $\eta \rightarrow 0$  is equivalent to letting  $Pr$  tend to infinity. The opposite order corresponds to the limit  $Pr \rightarrow 0$ . As first observed in ref. [8], when these limits are considered the range of weak compressibility splits into two ranges: what is now called the range of weak compressibility in the strict sense,  $0 \leq \wp < (d-2+\xi)/(2\xi)$ , and the range of *intermediate* compressibility,  $(d-2+\xi)/(2\xi) \leq \wp < d/\xi^2$ . In the former range, the order of the limits  $\kappa \rightarrow 0$  and  $\eta \rightarrow 0$  is not relevant for the Lagrangian dynamics [14]. At small scales, fluid particles disperse irrespective of the order of the limits, and therefore we expect the invariant measure of the separation to be non-degenerate. By contrast, the order matters in the latter range [14]. For intermediate values of the compressibility, if the viscous regularization is removed before the diffusivity ( $Pr \rightarrow 0$ ), the small-scale Lagrangian dynamics is once more characterized by the explosive separation of the trajectories. If  $\kappa$  goes to zero before  $\eta$  ( $Pr \rightarrow \infty$ ), the trajectories coalesce also at small scales,

and the invariant measure of the separation should degenerate into a Dirac delta function.

In summary, we believe that the present study captures the behavior of the Lagrangian trajectories on cylindrical manifolds for all  $Pr$  except for the limit  $Pr \rightarrow \infty$  in the intermediate-compressibility regime. These results are, moreover, relevant to turbulent transport of passive scalar fields in virtue of the relation subsisting between the scalar correlations and the dynamics of fluid particles [15, 11].

We conclude by noting that when both the dimensions of the plane are compactified one obtains the Kraichnan flow on a two-dimensional periodic square studied in ref. [4]. The velocity field considered there was however smooth in space.

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## A Covariance of velocity differences

For  $d = 2$ ,  $d' = 1$ ,  $\xi \in (0, 1)$ ,  $\xi' \in (1, 2)$ , the spatial covariance of the velocity field takes the form

$$D_{\alpha\beta}(\mathbf{r}) = \frac{1}{4\pi^2 L} \sum_{j=-\infty}^{\infty} e^{i\frac{j}{L}r_2} \int_{\mathbb{R}} dk_1 \frac{e^{ik_1 r_1} A_{\alpha\beta}((k_1, \frac{j}{L}); \wp)}{\left(k_1^2 + \frac{j^2}{L^2} + \frac{1}{\ell^2}\right)^{\frac{2+\xi}{2}}}$$

with  $\mathbf{r} \in \Omega$ . We first establish the convergence of the above series. It is convenient to denote the integrals by  $D_{\alpha\beta}^{(j)}(r_1)$  and thus rewrite the covariance as follows:

$$D_{\alpha\beta}(\mathbf{r}) = \frac{1}{4\pi^2 L} \sum_{j=-\infty}^{\infty} D_{\alpha\beta}^{(j)}(r_1) e^{i\frac{j}{L}r_2} = \frac{D_{\alpha\beta}^{(0)}(r_1)}{4\pi^2 L} + \frac{1}{2\pi^2 L} \sum_{j=1}^{\infty} D_{\alpha\beta}^{(j)}(r_1) \cos\left(\frac{j r_2}{L}\right). \quad (20)$$

Using the inequality

$$|A_{\alpha\beta}(\mathbf{k}; \wp)| \leq 1 - \wp + |2\wp - 1| \quad \forall \alpha, \beta = 1, 2 \quad \text{and} \quad \forall \mathbf{k} \in \mathbb{R} \times \frac{1}{L}\mathbb{Z},$$

we obtain that the coefficients of the series satisfy for all  $r_1$

$$|D_{\alpha\beta}^{(j)}(r_1)| \leq (1 - \wp + |2\wp - 1|) M_j$$

with (e.g., ref. [9], formula I 1.5(2))

$$M_j = \int_{\mathbb{R}} dk_1 \left(k_1^2 + \frac{j^2}{L^2} + \frac{1}{\ell^2}\right)^{-\frac{2+\xi}{2}} = \frac{\sqrt{\pi} \Gamma\left(\frac{1+\xi}{2}\right)}{\Gamma\left(1 + \frac{\xi}{2}\right)} \left(\frac{j^2}{L^2} + \frac{1}{\ell^2}\right)^{-\frac{1+\xi}{2}}.$$

The series  $\sum_{j=1}^{\infty} M_j$  converges for all  $\xi \in (0, 1)$  and  $\ell > 0$  as well as in the limit  $\ell \rightarrow \infty$ , as it can be checked by means of the integral test. Then, the Weierstrass criterion guarantees that the series in the right-hand-side of eq. (20) converge uniformly and absolutely on  $\Omega$ . The uniform convergence will allow us to compute  $\lim_{\ell \rightarrow \infty} d_{\alpha\beta}(\mathbf{r})$  by exchanging limit and summation.



The basic analytical ingredient to derive eqs. (7)–(9) is

$$\int_{-\infty}^{\infty} \frac{e^{ik_1 r_1}}{(k_1^2 + z^2)^{\nu+1/2}} dk_1 = \frac{2^{1-\nu} \pi^{1/2} K_\nu(|zr_1|) |r_1|^\nu}{\Gamma(\nu + \frac{1}{2}) |z|^\nu} \quad (21)$$

with  $\text{Re}(\nu) > -1/2$  and  $|\arg z| < \pi/2$  (e.g., ref. [9], formula II 7.12(27)).

We first compute the correlation of the axial component of the velocity; the correlation of the other components may be easily derived from  $D_{11}(\mathbf{r})$ .

In the limit  $\ell \rightarrow \infty$ , we have

$$\begin{aligned} \lim_{\ell \rightarrow \infty} D_{11}^{(j \neq 0)}(r_1) &= \int_{-\infty}^{\infty} dk_1 \frac{e^{ik_1 r_1} \left[ (1 - \wp) \frac{j^2}{L^2} + \wp k_1^2 \right]}{\left( k_1^2 + \frac{j^2}{L^2} \right)^{2+\frac{\xi}{2}}} \\ &= (1 - \wp) \frac{j^2}{L^2} \int_{-\infty}^{\infty} dk_1 \frac{e^{ik_1 r_1}}{\left( k_1^2 + \frac{j^2}{L^2} \right)^{2+\frac{\xi}{2}}} + \wp \int_{-\infty}^{\infty} dk_1 \frac{k_1^2 e^{ik_1 r_1}}{\left( k_1^2 + \frac{j^2}{L^2} \right)^{2+\frac{\xi}{2}}} \\ &= (1 - 2\wp) \frac{j^2}{L^2} \int_{-\infty}^{\infty} dk_1 \frac{e^{ik_1 r_1}}{\left( k_1^2 + \frac{j^2}{L^2} \right)^{2+\frac{\xi}{2}}} + \wp \int_{-\infty}^{\infty} dk_1 \frac{e^{ik_1 r_1}}{\left( k_1^2 + \frac{j^2}{L^2} \right)^{1+\frac{\xi}{2}}} \\ &= (1 - 2\wp) \left| \frac{j}{L} \right|^{\frac{1-\xi}{2}} \frac{\pi^{1/2} |r_1|^{(3+\xi)/2} K_{\frac{3+\xi}{2}} \left( \left| \frac{j}{L} r_1 \right| \right)}{2^{(1+\xi)/2} \Gamma(2 + \frac{\xi}{2})} \\ &\quad + \wp \left| \frac{j}{L} \right|^{-\frac{1+\xi}{2}} \frac{2^{(1-\xi)/2} \pi^{1/2} |r_1|^{(1+\xi)/2} K_{\frac{1+\xi}{2}} \left( \left| \frac{j}{L} r_1 \right| \right)}{\Gamma(1 + \frac{\xi}{2})}. \end{aligned}$$

To compute  $D_{11}^{(j)}(0)$  we can use the asymptotic expansion of  $K_\nu(z)$  for  $z \rightarrow 0$

$$K_\nu(x) \sim \frac{\Gamma(\nu)}{2} \left( \frac{x}{2} \right)^{-\nu} + \frac{\Gamma(-\nu)}{2} \left( \frac{x}{2} \right)^\nu + O(x^{2-\nu}) \quad (\nu < 1) \quad (22)$$

$$K_\nu(x) \sim \frac{\Gamma(\nu)}{2} \left( \frac{x}{2} \right)^{-\nu} - \frac{\Gamma(\nu)}{2(\nu-1)} \left( \frac{x}{2} \right)^{2-\nu} + O(x^\nu) \quad (1 < \nu < 2). \quad (23)$$

Hence we obtain

$$\begin{aligned} \lim_{\ell \rightarrow \infty} D_{11}^{(j \neq 0)}(0) &= \wp \left| \frac{j}{L} \right|^{-1-\xi} \frac{\pi^{1/2} \Gamma(1+\frac{\xi}{2})}{\Gamma(1+\frac{\xi}{2})} + (1 - 2\wp) \left| \frac{j}{L} \right|^{-1-\xi} \frac{\pi^{1/2} \Gamma(3+\frac{\xi}{2})}{\Gamma(2+\frac{\xi}{2})} \\ &= \left| \frac{j}{L} \right|^{-1-\xi} \frac{\pi^{1/2} (1 + \xi - \wp \xi) \Gamma(1+\frac{\xi}{2})}{2\Gamma(2+\frac{\xi}{2})}. \end{aligned}$$

For  $j = 0$  and  $\ell < \infty$ , we have

$$D_{11}^{(0)}(r_1) = \wp \int_{-\infty}^{\infty} dk_1 \frac{e^{ik_1 r_1}}{(k_1^2 + \ell^{-2})^{2+\frac{\xi}{2}}} = \wp \frac{2^{\frac{1-\xi}{2}} \pi^{1/2}}{\Gamma(1+\frac{\xi}{2})} |\ell r_1|^{\frac{1+\xi}{2}} K_{\frac{1+\xi}{2}} \left( \left| \frac{r_1}{\ell} \right| \right).$$

The asymptotic expansion (22) shows that  $D_{11}^{(0)}(0)$  diverges like  $\ell^{1+\xi}$  as  $\ell \rightarrow \infty$ . By using expansion (22), it is nonetheless possible to show that

$$\lim_{\ell \rightarrow \infty} [D_{11}^{(0)}(0) - D_{11}^{(0)}(r_1)] = \wp \frac{\pi^{1/2} \left| \Gamma(-\frac{1+\xi}{2}) \right|}{2^{1+\xi} \Gamma(1+\frac{\xi}{2})} |r_1|^{1+\xi}$$

(note that  $\Gamma(-(1+\xi)/2) < 0$  for  $0 < \xi < 1$ ). Hence, the covariance of the axial component of the velocity difference,  $d_{11}(\mathbf{r})$ , has a finite limit as  $\ell \rightarrow \infty$ .

For the other components we have:

$$D_{22}^{(j)}(r_1) = \int_{-\infty}^{\infty} dk_1 \frac{e^{ik_1 r_1} \left[ (1-\wp)k_1^2 + \wp \frac{j^2}{L^2} \right]}{\left( k_1^2 + \frac{j^2}{L^2} \right) \left( k_1^2 + \frac{j^2}{L^2} + \frac{1}{\ell^2} \right)^{\frac{2+\xi}{2}}}$$

and

$$D_{12}^{(j)}(r_1) = D_{21}^{(j)}(r_1) = (2\wp - 1) \frac{j}{L} \int_{-\infty}^{\infty} dk_1 \frac{k_1 e^{ik_1 r_1}}{\left( k_1^2 + \frac{j^2}{L^2} \right) \left( k_1^2 + \frac{j^2}{L^2} + \frac{1}{\ell^2} \right)^{\frac{2+\xi}{2}}}.$$

Therefore,  $D_{22}^{(j)}(r_1)$  can be derived from  $D_{11}^{(j)}(r_1)$  by replacing  $\wp$  with  $1 - \wp$ :

$$\begin{aligned} \lim_{\ell \rightarrow \infty} D_{22}^{(j \neq 0)}(r_1) &= (2\wp - 1) \left| \frac{j}{L} \right|^{\frac{1-\xi}{2}} \frac{\pi^{1/2} K_{\frac{3+\xi}{2}} \left( \left| \frac{j}{L} r_1 \right| \right) |r_1|^{(3+\xi)/2}}{2^{(1+\xi)/2} \Gamma\left(2 + \frac{\xi}{2}\right)} \\ &\quad + (1 - \wp) \left| \frac{j}{L} \right|^{-\frac{1+\xi}{2}} \frac{2^{(1-\xi)/2} \pi^{1/2} K_{\frac{1+\xi}{2}} \left( \left| \frac{j}{L} r_1 \right| \right) |r_1|^{(1+\xi)/2}}{\Gamma\left(1 + \frac{\xi}{2}\right)}, \end{aligned}$$

$$\lim_{\ell \rightarrow \infty} D_{22}^{(j \neq 0)}(0) = \left| \frac{j}{L} \right|^{-1-\xi} \frac{\sqrt{\pi} (1 + \wp \xi) \Gamma\left(\frac{1+\xi}{2}\right)}{2 \Gamma\left(2 + \frac{\xi}{2}\right)},$$

$$\lim_{\ell \rightarrow \infty} [D_{22}^{(0)}(0) - D_{22}^{(0)}(r_1)] = (1 - \wp) \frac{\pi^{1/2} \left| \Gamma\left(-\frac{1+\xi}{2}\right) \right|}{2^{1+\xi} \Gamma\left(1 + \frac{\xi}{2}\right)} |r_1|^{1+\xi}.$$

The mixed correlation can be obtained, for  $j \neq 0$ , by differentiating formula (21) with respect to  $r_1$  and by using  $\frac{d}{dx}[x^\nu K_\nu(x)] = -x^\nu K_{\nu-1}(x)$  (e.g., ref [9], formula II 7.11(21)):

$$\lim_{\ell \rightarrow \infty} D_{12}^{(j \neq 0)}(r_1) = i(2\wp - 1) \frac{j}{L} \left| \frac{j}{L} \right|^{-\frac{1+\xi}{2}} \frac{\sqrt{\pi} |r_1|^{(3+\xi)/2}}{2^{\frac{1+\xi}{2}} \Gamma\left(2 + \frac{\xi}{2}\right)} K_{\frac{1+\xi}{2}} \left( \left| \frac{j}{L} r_1 \right| \right) \text{sgn}(r_1).$$

Hence

$$\lim_{\ell \rightarrow \infty} D_{12}^{(j \neq 0)}(0) = 0.$$

For  $j = 0$  we have

$$D_{12}^{(0)}(r_1) = 0 \quad \forall r_1 \in \mathbb{R}.$$

Finally, eqs. (7)–(9) may be derived by recalling that

$$d_{\alpha\beta}(\mathbf{r}) = \frac{1}{4\pi^2 L} \sum_{j=-\infty}^{\infty} \left[ D_{\alpha\beta}^{(j)}(0) - D_{\alpha\beta}^{(j)}(r_1) e^{i \frac{j}{L} r_2} \right]$$

and using the uniform convergence of the series in the right-hand-side of eq. (20).

## B Proof of Lemma 1

To prove property 3 of lemma 1, we first observe that

$$\mathbb{E}(\|\mathbf{R}(t)\|^2) = \|\mathbf{R}(0)\|^2 + 2\mathbb{E}\left(\int_0^t \|\sigma(\mathbf{R}(s))\|^2 ds\right) \leq C_2 \left(1 + \int_0^t \mathbb{E}(\|\mathbf{R}(s)\|^2) ds\right),$$

where  $C_2 > 0$  and  $\|\sigma\| := [\text{Tr}(\sigma\sigma^T)]^{1/2}$ . The above inequality is a consequence of (15). Gronwall's inequality then yields

$$\mathbb{E}(\|\mathbf{R}(t)\|^2) \leq C_2 e^{C_2 t} \quad (24)$$

for all  $t > 0$ .

Following ref. [26], we consider, for all  $\gamma > 0$ , the function  $\varphi_\gamma : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  with

$$\varphi_\gamma(z) = \begin{cases} z & 0 \leq z \leq \gamma \\ \varphi_\gamma(\gamma + 1) & \gamma + 1 \leq z \end{cases}$$

and  $\varphi_\gamma \in \mathcal{C}^2(\mathbb{R}_+)$  and monotonically non-decreasing. Moreover, we define  $V_\gamma := \varphi_\gamma \circ V$ . Applying  $\mathcal{A}$  to  $V_\gamma$  and taking into account (10) yield

$$|\mathcal{A}V_\gamma(\mathbf{r})| = \sigma_{11}^2(\mathbf{r}) \left| \varphi'_\gamma(V(\mathbf{r})) \frac{\partial^2 V}{\partial r_1^2} + \varphi''_\gamma(V(\mathbf{r})) \left( \frac{\partial V}{\partial r_1} \right)^2 \right| = O(|r_1|^{3+\xi-4h}) \quad (25)$$

with  $3 + \xi - 4h < 2$  as  $\|\mathbf{r}\| \rightarrow \infty$ .

Since  $V_\gamma \in \mathcal{C}_b^2(\Omega)$ , we have for all  $\gamma > 0$

$$T_t V_\gamma(\mathbf{r}) = V_\gamma(\mathbf{r}) + \int_0^t T_s(\mathcal{A}V_\gamma)(\mathbf{r}) ds.$$

We now show that each term of the above equation tends as  $\gamma \rightarrow \infty$  to the corresponding term in property 3 of lemma 1.

Obviously,  $\lim_{\gamma \rightarrow \infty} V_\gamma(\mathbf{r}) = V(\mathbf{r})$  for all  $\mathbf{r} \in \Omega$ . Likewise,  $V_\gamma(\mathbf{R}(t)) \nearrow V(\mathbf{R}(t))$  almost everywhere as  $\gamma \rightarrow \infty$ . Therefore, by the monotone convergence theorem  $\lim_{\gamma \rightarrow \infty} T_t V_\gamma(\mathbf{r}) = T_t V(\mathbf{r})$ . Finally,  $\lim_{\gamma \rightarrow \infty} \mathcal{A}V_\gamma(\mathbf{R}(t)) = \mathcal{A}V(\mathbf{R}(t))$  and, from eqs. (25) and (24),  $|\mathcal{A}V_\gamma(\mathbf{R}(t))| \leq C_3(1 + \|\mathbf{R}(t)\|^2)$  with  $C_3 > 0$  and  $\mathbb{E}(\|\mathbf{R}(t)\|^2) < \infty$ . Then, it follows from the bounded convergence theorem that

$$\lim_{\gamma \rightarrow \infty} \int_0^t T_s(\mathcal{A}V_\gamma)(\mathbf{r}) ds = \int_0^t T_s(\mathcal{A}V)(\mathbf{r}) ds.$$

This concludes the proof.

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